THE REARRANGEMENT INEQUALITY

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We will introduce our subject via an example, taken from a Chinese competition in 1978.

"Ten people queue up before a tap to fill their buckets. Each bucket requires a different time to fill. In what order should the people queue up so as to minimize their combined waiting time?"

Common sense suggests that they queue up in ascending order of "bucket-filling time". Let us see if our intuition leads us astray. We will denote by $T_1 < T_2 < \cdots < T_{10}$ the times required to fill the respective buckets.

If the people queue up in the order suggested, their combined waiting time will be given by $T = 10T_1 + 9T_2 + \cdots + T_{10}$. For a different queueing order, the combined waiting time will be $S = 10S_1 + 9S_2 + \cdots + S_{10}$, where $(S_1, S_2, \ldots, S_{10})$ is a permutation of $(T_1, T_2, \ldots, T_{10})$.

The two 10-tuples being different, there is a smallest index i for which $S_i \neq T_i$. Then $S_j = T_i < S_i$ for some $j > i$. Define $S'_i = S_j, S'_j = S_i$ and $S'_k = S_k$ for $k \neq i, j$. Let $S' = 10S'_1 + 9S'_2 + \cdots + S'_{10}$. Then

$$S - S' = (11 - i)(S_i - S'_i) + (11 - j)(S_j - S'_j) = (S_i - S_j)(j - i) > 0.$$

Hence the switching results in a lower combined waiting time.

If $(S'_1, S'_2, \ldots, S'_{10}) \neq (T_1, T_2, \ldots, T_{10})$, this switching process can be repeated again. We will reach $(T_1, T_2, \ldots, T_{10})$ in at most 9 steps. Since the combined waiting time is reduced in each step, T is indeed the minimum combined waiting time.

We can generalize this example to the following result.

The Rearrangement Inequality.
Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(a'_1, a'_2, \ldots, a'_n)$ of $(a_1, a_2, \ldots, a_n)$, we have

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geq a'_1b_1 + a'_2b_2 + \cdots + a'_nb_n \\ &\geq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n, \end{aligned}$$

with equality if and only if $(a'_1, a'_2, \ldots, a'_n)$ is equal to $(a_1, a_2, \ldots, a_n)$ or $(a_n, a_{n-1}, \ldots, a_1)$ respectively.

This can be proved by the switching process used in the introductory example. See for instance [1] or [2], which contain more general results. Note that unlike many inequalities, we do not require the numbers involved to be positive.

Corollary 1.
Let $a_1, a_2, \ldots, a_n$ be real numbers and $(a'_1, a'_2, \ldots, a'_n)$ be a permutation of $(a_1, a_2, \ldots, a_n)$. Then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1a'_1 + a_2a'_2 + \cdots + a_na'_n.$$

Corollary 2.
Let $a_1, a_2, \ldots, a_n$ be positive numbers and $(a'_1, a'_2, \ldots, a'_n)$ be a permutation of $(a_1, a_2, \ldots, a_n)$. Then

$$\frac{a'_1}{a_1} + \frac{a'_2}{a_2} + \cdots + \frac{a'_n}{a_n} \geq n.$$

A 1935 Kürschák problem in Hungary asked for the proof of Corollary 2, and a 1940 Moscow Olympiad problem asked for the proof of the special case $(a'_1, a'_2, \ldots, a'_n) = (a_2, a_3, \ldots, a_n, a_1)$.

We now illustrate the power of the Rearrangement Inequality by giving simple solutions to a number of competition problems.

Example 1. (International Mathematical Olympiad, 1975)
Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be real numbers. Let $(z_1, z_2, \cdots, z_n)$ be a permutation of $(y_1, y_2, \ldots, y_n)$. Prove that

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 \leq (x_1 - z_1)^2 + (x_2 - z_2)^2 + \cdots + (x_n - z_n)^2.$$

Solution:
Note that we have $y_1^2 + y_2^2 + \cdots + y_n^2 = z_1^2 + z_2^2 + \cdots + z_n^2$. After expansion and simplification, the desired inequality is equivalent to

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \geq x_1 z_1 + x_2 z_2 + \cdots + x_n z_n,$$

which follows from the Rearrangement Inequality.

Example 2. (International Mathematical Olympiad, 1978)
Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution:
Let $(a'_1, a'_2, \ldots, a'_n)$ be the permutation of $(a_1, a_2, \ldots, a_n)$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$. Then $a'_i \geq i$ for $1 \leq i \leq n$. By the Rearrangement Inequality,

$$\begin{aligned} \frac{a_1}{1^2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} &\geq \frac{a'_1}{1^2} + \frac{a'_2}{2^2} + \cdots + \frac{a'_n}{n^2} \\ &\geq \frac{1}{1^2} + \frac{2}{2^2} + \cdots + \frac{n}{n^2} \\ &\geq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}. \end{aligned}$$

Example 3. (International Mathematical Olympiad, 1964)
Let a, b and c be the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

Solution:
We may assume that $a \geq b \geq c$. We first prove that $c(a+b-c) \geq b(c+a-b) \geq a(b+c-a)$. Note that $c(a+b-c) - b(c+a-b) = (b-c)(b+c-a) \geq 0$. The second inequality can be proved in the same manner. By the Rearrangement Inequality, we have

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq ba(b+c-a) + cb(c+a-b) + ac(a+b-c),$$

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq ca(b+c-a) + ab(c+a-b) + bc(a+b-c).$$

Adding these two inequalities, the right side simplifies to $6abc$. The desired inequality now follows.

Example 4. (International Mathematical Olympiad, 1983)
Let a, b and c be the sides of a triangle. Prove that $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$.

Solution:
We may assume that $a \geq b, c$. If $a \geq b \geq c$, we have $a(b+c-a) \leq b(c+a-b) \leq c(a+b-c)$ as in Example 3. By the Rearrangement Inequality,

$$\begin{aligned} & \frac{1}{c}a(b+c-a) + \frac{1}{a}b(c+a-b) + \frac{1}{b}c(a+b-c) \\ \leq & \frac{1}{a}a(b+c-a) + \frac{1}{b}b(c+a-b) + \frac{1}{c}c(a+b-c) \\ = & a+b+c. \end{aligned}$$

This simplifies to $\frac{1}{c}a(b-a) + \frac{1}{a}b(c-b) + \frac{1}{b}c(a-c) \leq 0$, which is equivalent to the desired inequality. If $a \geq c \geq b$, then $a(b+c-a) \leq c(a+b-c) \leq b(c+a-b)$. All we have to do is interchange the second and the third terms of the displayed lines above.

Simple as it sounds, the Rearrangement Inequality is a result of fundamental importance. We shall derive from it many familiar and useful inequalities.

Example 5. The Arithmetic Mean Geometric Mean Inequality.
Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof:
Let $G = \sqrt[n]{x_1 x_2 \cdots x_n}$, $a_1 = \dfrac{x_1}{G}, a_2 = \dfrac{x_1 x_2}{G^2}, \ldots, a_n = \dfrac{x_1 x_2 \cdots x_n}{G^n} = 1$. By Corollary 2,

$$n \leq \frac{a_1}{a_n} + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_{n-1}} = \frac{x_1}{G} + \frac{x_2}{G} + \cdots + \frac{x_n}{G},$$

which is equivalent to $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \geq G$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$, or $x_1 = x_2 = \cdots = x_n$.

Example 6. The Geometric mean Harmonic Mean Inequality.
Let $x_1, x_2, \ldots, x_n$ be positive numbers. Then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof:
Let $G, a_1, a_2, \ldots, a_n$ be as in Example 5. By Corollary 2,

$$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \cdots + \frac{G}{x_n},$$

which is equivalent to

$$G \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Example 7. The Root Mean Square Arithmetic Mean Inequality.
Let $x_1, x_2, \ldots, x_n$ be real numbers. Then

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof:
By Corollary 1, we have

$$\begin{array}{rcl} x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1x_2 + x_2x_3 + \cdots + x_nx_1, \\ x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1x_3 + x_2x_4 + \cdots + x_nx_2, \\ \cdots & \geq & \cdots \\ x_1^2 + x_2^2 + \cdots + x_n^2 & \geq & x_1x_n + x_2x_1 + \cdots + x_nx_{n-1}. \end{array}$$

Adding these and $x_1^2 + x_2^2 + \cdots + x_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, we have

$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n^2)^2,$$

which is equivalent to the desired result. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Example 8. Cauchy's Inequality.
Let $a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n$ be real numbers. Then

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2),$$

with equality if and only if for some constant k, $a_i = kb_i$ for $1 \leq i \leq n$ or $b_i = ka_i$ for $1 \leq i \leq n$.

Proof:
If $a_1 = a_2 = \cdots = a_n = 0$ or $b_1 = b_2 = \cdots = b_n = 0$, the result is trivial. Otherwise, define $S = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ and $T = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$. Since both are non-zero, we may let $x_i = \dfrac{a_i}{S}$ and $x_{n+i} = \dfrac{b_i}{T}$ for $1 \leq i \leq n$. By Corollary 1,

$$\begin{array}{rcl} 2 & = & \dfrac{a_1^2 + a_2^2 + \cdots + a_n^2}{S^2} + \dfrac{b_1^2 + b_2^2 + \cdots + b_n^2}{T^2} \\ & = & x_1^2 + x_2^2 + \cdots + x_{2n}^2 \\ & \geq & x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n} + x_{n+1}x_1 + x_{n+2}x_2 + \cdots + x_{2n}x_n \\ & = & \dfrac{2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)}{ST}, \end{array}$$

which is equivalent to the desired result. Equality holds if and only if $x_i = x_{n+i}$ for $1 \leq i \leq n$, or $a_iT = b_iS$ for $1 \leq i \leq n$.

We shall conclude this paper with two more examples whose solutions are left as exercises.

Example 9. (Chinese competition, 1984) Prove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_n^2}{x_1} \geq x_1 + x_2 + \cdots + x_n$$

for all positive numbers $x_1, x_2, \ldots, x_n$.

Example 10. (Moscow Olympiad, 1963) Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all positive numbers a, b and c.

References:

1. G. Hardy, J. Littlewood and G. Polya, "Inequalities", Cambridge University Press, Cambridge, paperback edition, (1988) 260-299.

2. K. Wu, The Rearrangement Inequality, Chapter 8 in "Lecture Notes in Mathematics Competitions and Enrichment for High Schools" (in Chinese), ed. K. Wu et al., (1989) 8:1-8:25.