Coloring Cubic Graphs by Steiner Triple Systems

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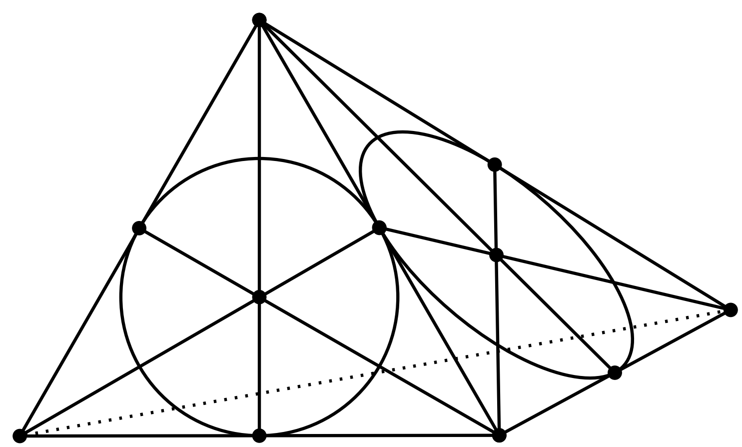

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Ordinary Edge Colorings

- Let G be a cubic graph.
- Using some set of colors we color its egdes.
- The colors of the edges meeting at a vertex must be all distinct.

Theorem 1 (Vizing, 1964). *Cubic graph can be colored by either 3 or 4 colors.*

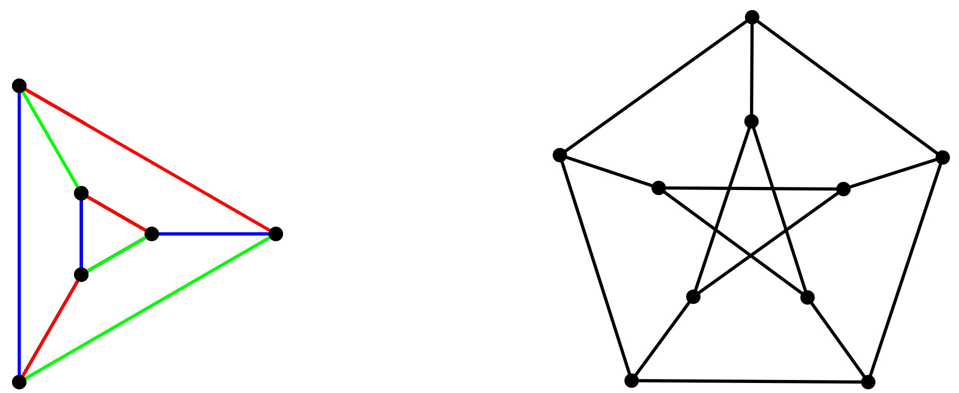

Figure 1: Three-sided prism and the Petersen graph

Graphs that need 4 colors are called *Snarks*. Snarks include all graphs with a bridge, Petersen graph and infinitely many others.

Steiner Triple Systems

- We would like somehow to extend ordinary 3-colorings, to be able to color (some) Snarks.
- We allow more than one triple of colors of egdes at a vertex.
- Two colors should determine third.

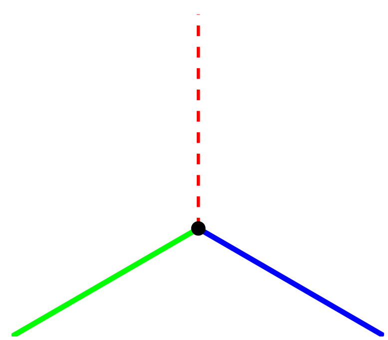

Figure 2: Green edge and blue edge imply color of the third red edge.

Definition 1. Steiner triple system *(STS for short)* $\mathcal{S}$ *is a tuple* $\mathcal{S} = (P, T)$*.* P *is set of finite set of points (contaning at least three elements).* T *is system of 3-element-subsets of points such that each pair of points is contained in exactly one set of* T*.*

Examples of Steiner Triple Systems

- Trivial STS $= (\{1,2,3\}, \{\{1,2,3\}\})$.
- Fano plane $\mathcal{F} = PG(2,2) = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 - \{0,0,0\}, \{\{x,y,z\} \mid x+y+z=0\})$

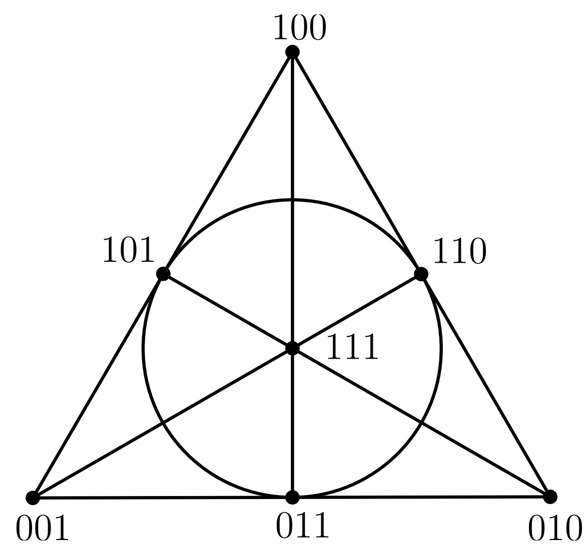

Figure 3: Fano plane

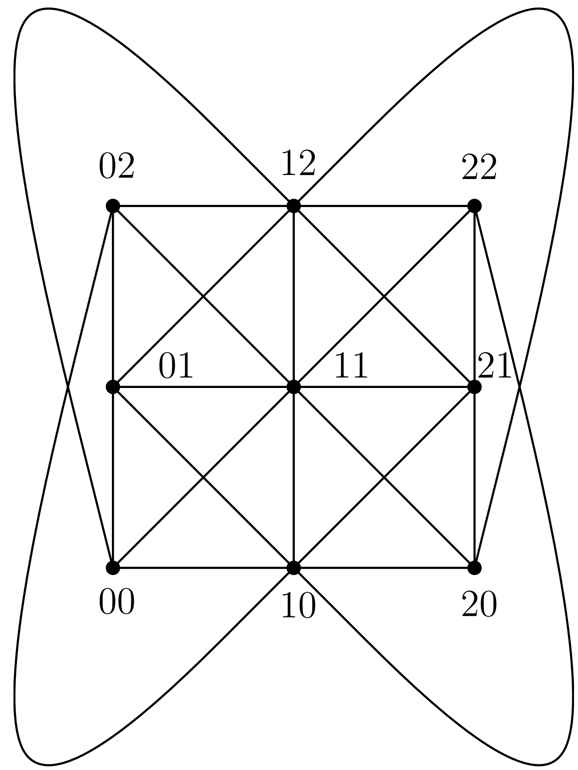

Figure 4: Affine plane AG(2,3)

Affine plane $AG(2,3) = (\mathbb{Z}_3 \times \mathbb{Z}_3, \{\{x, y, z\} \mid x + y + z = 0\})$

Configurations in STS

Combinatorial designers deal also with configurations in Steiner triple systems. For example C_{15} can be found in every nontrivial system.

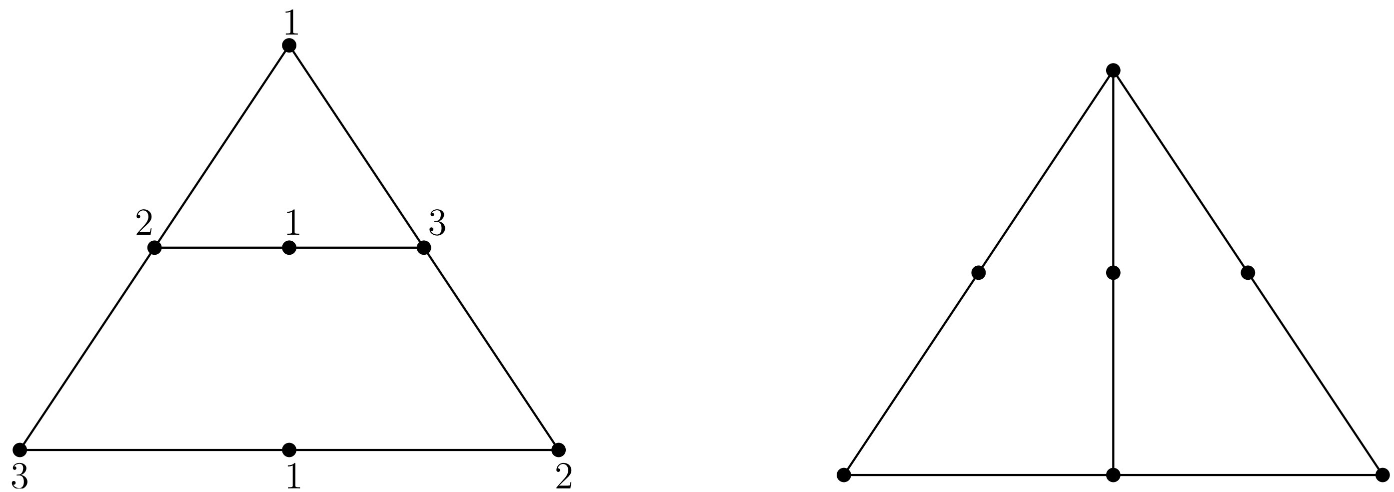

Figure 5: Cofingurations C_{14} and C_{15}.

Steiner Colorings

Definition 2. *Let $G = (V, E)$ be cubic graph and let $\mathcal{S} = (P, T)$ be Steiner triple system.* Steiner coloring *is a map $\phi : E \longrightarrow P$ such that colors of the three incident edges at each vertex form a triple of $\mathcal{S}$.*

The general question is: "Which systems color which graphs?".

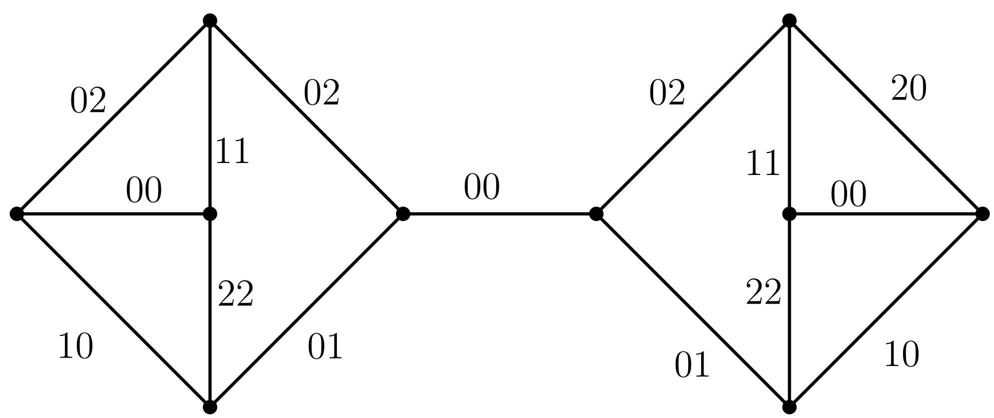

Figure 6: Graph with a bridge colored by the affine plane AG(2,3)

Known Results and Open Problems

Known results

- There exists a system of order 381 that colors all the cubic graphs.
- Every nontrivial system colors all bridgeless graphs.
- Projective systems cannot color graphs with bridges.
- Decision problem "Is given graph is a Snark?" is NP-complete.

Open problems

- What is the smallest system that colors everything?
- What can be colored by affine systems?
- Minimize the number of triples used in a coloring.
- What configurations can color what?
- Find effective (i.e. polynomial) coloring algorithms.

Connection with Perfect Matchings

Theorem 2 (Petersen, 1891). *Every bridgeless cubic graph has a perfect matching.*

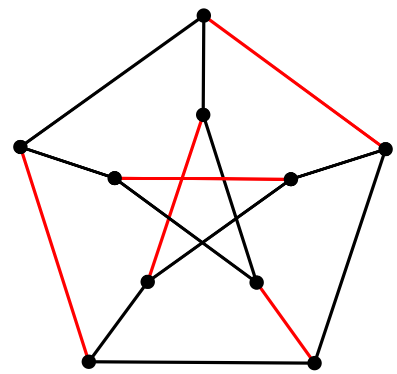

Figure 7: Perfect matching in the Petersen graph

Conjecture 1 (Fan-Raspaud, 1994). *Every bridgeless cubic graph contains three perfect matchings with empty intersection.*

Conjecture 2 (Berge-Fulkerson). *Every bridgeless cubic graph contains six perfect matchings such that each edge appears in exactly two of them.*

Both these conjectures can be formulated in terms of Steiner colorings (or coloring by configurations).